UNCERTAINTY MODELING FOR EFFICIENT VISUAL ODOMETRY VIA INERTIAL SENSORS ON MOBILE DEVICES

SUPPLEMENTARY MATERIAL

COMPUTATION OF METRIC TRANSLATIONAL VELOCITY AND CORRESPONDING UNCERTAINTY USING INERTIAL SENSORS AND VISUAL TRACKING

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ABSTRACT

We formulate estimation of metric velocity using a visually tracked point, accelerometer reading and gyroscope readings that are in sync with video rate, in parallel with [1]. Uncertainty on the estimated velocity is also derived via the uncertainties on the utilized measurements.

1. INTRODUCTION

In order to compute the change in position between two time instances, in addition to the accelerometer readings, translational velocity of the camera is needed. In this document, we present a method to estimate the instantaneous velocity.

Kneip *et al.* [1] formulated the computation of metric translational velocity from accelerometer measurements and a visually tracked point in three frames. In this report, we adopt their formulation with several small differences in order to derive the equation relating camera velocity with inertial and visual measurements. Following the formulation, computation of the covariance matrix of the estimated velocity is presented.

In [1], the sample rate of the accelerometer is assumed to be much greater than frame rate of the image stream and a recursive formulation is developed for integration of accelerometer readings. Since two rates are similar in the utilized inertial measurement units on ASUS TF201 tablet device (48 Hz accelerometer and 30 Hz image stream), for a reasonable formulation, we assume that the inertial readings are taken simultaneously with each image and resample inertial readings at the video rate.

2. TRANSLATIONAL VELOCITY ESTIMATION

As it can be seen in Figure 1, three poses of the camera, $\vec{\phi}_n$, $\vec{\phi}_{n-1}$ and $\vec{\phi}_{n-2}$ at time instants t_n, t_{n-1} and t_{n-2}, are considered. Time intervals are defined as $t_1 = t_n - t_{n-1}$, $t_2 = t_n - t_{n-2}$ and $t_3 = t_{n-1} - t_{n-2}$. Accelerometer readings $\vec{\alpha}_{n-1}$ and $\vec{\alpha}_{n-2}$ are retrieved at t_{n-1} and t_{n-2}. Instantaneous velocities are denoted as $\vec{v}_n$, $\vec{v}_{n-1}$ and $\vec{v}_{n-2}$. Accelerations and velocities are represented in the coordinate frames corresponding to the poses at their time instants. The rotation from $\vec{\phi}_{n-1}$ to $\vec{\phi}_n$ is defined as $\tilde{q}_1$, from $\vec{\phi}_{n-2}$ to $\vec{\phi}_n$ as $\tilde{q}_2$ and from $\vec{\phi}_{n-2}$ to $\vec{\phi}_{n-1}$ as $\tilde{q}_3$, *i.e.* $\tilde{q}_2 = \tilde{q}_1\tilde{q}_3$, where $\tilde{}$ represents quaternion representation. These rotation quaternions are assumed to be known and can be estimated from visual measurements or gyroscope readings. The translation from $\vec{\phi}_n$ to $\vec{\phi}_{n-1}$ is defined as $\vec{\tau}_1$, from $\vec{\phi}_n$ to $\vec{\phi}_{n-2}$ as $\vec{\tau}_2$ and from $\vec{\phi}_{n-1}$ to $\vec{\phi}_{n-2}$ as $\vec{\tau}_3$. Translations are defined in the latest coordinate frame.

We use the vector and quaternion representation of the variables interchangeably such that:

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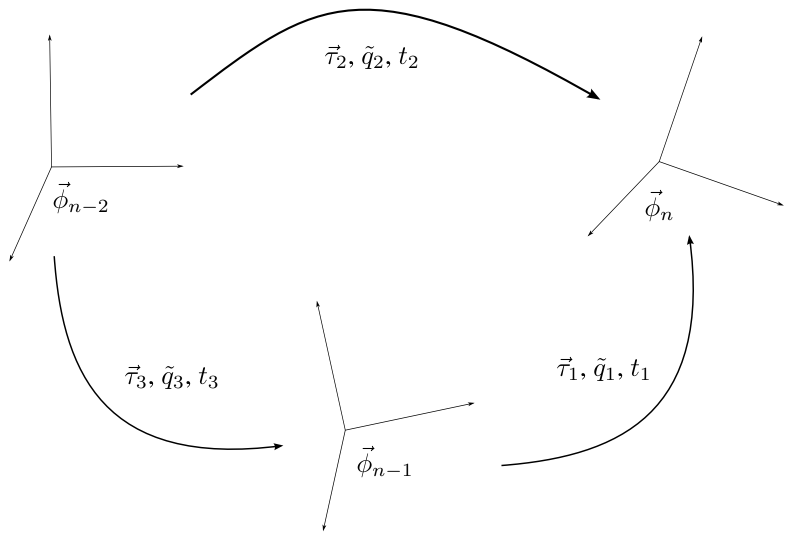

Fig. 1: Definition of three poses and relative transformations

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \vec{\alpha} \Leftrightarrow \tilde{\alpha} = \alpha_1 i + \alpha_2 j + \alpha_3 k \tag{1}$$

$$\begin{bmatrix}q_s\\ q_a\\ q_b\\ q_c\end{bmatrix} = \vec{q} \Leftrightarrow \tilde{q} = q_s + q_a i + q_b j + q_c k \tag{2}$$

We want to determine $\vec{v}_n$ using the accelerations $\vec{a}_{n-1}$ and $\vec{a}_{n-2}$, locations $\vec{x}_n$, $\vec{x}_{n-1}$ and $\vec{x}_{n-2}$ of a tracked point in three frames and attitudes $\tilde{q}_n$, $\tilde{q}_{n-1}$ and $\tilde{q}_{n-2}$ of three frames. Let us start by representing $\vec{v}_{n-1}$ using the known and target variables.

$$\vec{v}_{n-1} = \tilde{q}_1^* \tilde{v}_n \tilde{q}_1 - \vec{a}_{(n-1)} t_1 \tag{3}$$

We can represent $\vec{v}_{n-1}$ in the latest coordinate frame as $\vec{v}_{(n-1|n)}$ such that $\vec{v}_{(n-1|n)} = \tilde{q}\tilde{v}_{n-1}\tilde{q}^*$.

$$\vec{v}_{(n-1|n)} = \vec{v}_n - \tilde{q}_1 \tilde{a}_{n-1} \tilde{q}_1^* t_1 \tag{4a}$$

$$= \vec{v}_n - \vec{a}_{(n-1|n)} t_1 \tag{4b}$$

The translation $\vec{\tau}_{(1|n-1)}$ can be computed as:

$$\vec{\tau}_{(1|n-1)} = \vec{v}_{n-1} t_1 + \frac{1}{2} \vec{a}_{n-1} t_1^2 \tag{5}$$

Then $\vec{\tau}_1$ becomes:

$$\vec{\tau}_1 = \vec{v}_{(n-1|n)} t_1 + \frac{1}{2} \vec{a}_{(n-1|n)} t_1^2 \tag{6}$$

By substituting (4b) in (6), we get [1]:

$$\vec{\tau}_1 = \vec{v}_n t_1 - \vec{a}_{(n-1|n)} t_1^2 + \frac{1}{2} \vec{a}_{(n-1|n)} t_1^2 \tag{7a}$$

$$= \vec{v}_n t_1 - \frac{1}{2} \vec{a}_{(n-1|n)} t_1^2 \tag{7b}$$

Similarly, $\vec{\tau}_{(3|n-1)}$ is:

$$\vec{\tau}_{(3|n-1)} = \vec{v}_{n-1}t_3 - \frac{1}{2}\vec{a}_{(n-2|n-1)}t_3^2 \tag{8}$$

$$\begin{align} \vec{\tau}_3 &= \tilde{q}_1\tilde{\tau}_{(3|n-1)}\tilde{q}_1^* \tag{9a}\\ &= \vec{v}_{(n-1|n)}t_3 - \frac{1}{2}\vec{a}_{(n-2|n)}t_3^2 \tag{9b}\\ &= \vec{v}_n t_3 - \vec{a}_{(n-1|n)}t_1t_3 - \frac{1}{2}\vec{a}_{(n-2|n)}t_3^2 \tag{9c}\end{align}$$

We can then compute $\vec{\tau}_2$ as:

$$\begin{align} \vec{\tau}_2 &= \vec{\tau}_1 + \vec{\tau}_3 \tag{10a}\\ &= \vec{v}_n(t_1 + t_3) - \vec{a}_{(n-1|n)}(\frac{1}{2}t_1^2 + t_1t_3) - \vec{a}_{(n-2|n)}(\frac{1}{2}t_3^2) \tag{10b}\\ &= \vec{v}_n t_2 - \vec{a}_{(n-1|n)}(\frac{1}{2}t_1^2 + t_1t_3) - \vec{a}_{(n-2|n)}(\frac{1}{2}t_3^2) \tag{10c}\end{align}$$

When we define $\vec{\eta}_1$ and $\vec{\eta}_2$ as [1]:

$$\begin{align} \vec{\eta}_1 &= -\frac{1}{2}\vec{a}_{(n-1|n)}t_1^2 \tag{11a}\\ \vec{\eta}_2 &= -\vec{a}_{(n-1|n)}(\frac{1}{2}t_1^2 + t_1t_3) - \vec{a}_{(n-2|n)}(\frac{1}{2}t_3^2) \tag{11b}\end{align}$$

$\vec{\tau}_1$ and $\vec{\tau}_2$ simply becomes:

$$\begin{align} \vec{\tau}_1 &= \vec{v}_n t_1 + \vec{\eta}_1 \tag{12a}\\ \vec{\tau}_2 &= \vec{v}_n t_2 + \vec{\eta}_2 \tag{12b}\end{align}$$

We now have the translation between three frames in terms of the known variables and the desired velocity vector. Now, in order to relate the kinematic equations with the tracked interest point, we will relate the point locations in three frames with each other. Firstly, let us define *normalized image coordinates* as:

$$z'\vec{x}' = z'\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = K^{-1}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{13}$$

where K is the camera calibration matrix and x and y are the regular image coordinates. This representation of point locations is called *normalized* because the calibration matrix corresponding to the new coordinates is an identity matrix.

We can relate the normalized coordinates of the point at n^{th} and $(n-1)^{th}$ frames as:

$$z_{n-1}\vec{x}'_{n-1} = \tilde{q}_1^*(z_n\tilde{x}'_n + \tilde{\tau}_1)\tilde{q}_1 \tag{14}$$

When we plug in the value of $\vec{\tau}_1$, (14) becomes:

$$z_{n-1}\vec{x}'_{n-1} = \tilde{q}_1^*(z_n\tilde{x}'_n + \tilde{v}_n t_1 + \tilde{\eta}_1)\tilde{q}_1 \tag{15}$$

Let us separate three components of the velocity vector using quaternions $\tilde{q}_x = i$, $\tilde{q}_y = j$ and $\tilde{q}_z = k$ [1].

$$\begin{align} z_{n-1}\vec{x}'_{n-1} &= \tilde{q}_1^*(z_n\tilde{x}'_n + \tilde{q}_x v_x t_1 + \tilde{q}_y v_y t_1 + \tilde{q}_z v_z t_1 + \tilde{\eta}_1)\tilde{q}_1 \tag{16a}\\ &= z_n\tilde{q}_1^*\tilde{x}'_n\tilde{q}_1 + v_x t_1\tilde{q}_1^*\tilde{q}_x\tilde{q}_1 + v_y t_1\tilde{q}_1^*\tilde{q}_y\tilde{q}_1 + v_z t_1\tilde{q}_1^*\tilde{q}_z\tilde{q}_1 + \tilde{q}_1^*\tilde{\eta}_1\tilde{q}_1 \tag{16b}\end{align}$$

Let us represent the variables that are rotated by $\tilde{q}_1^*$ with 1^* as an extra subscript, such that $\tilde{q}_1^*\tilde{x}'_n\tilde{q}_1 = \tilde{x}'_{n|1^*}$. Then, the last equation becomes:

$$z_{n-1}\vec{x}'_{n-1} = z_{n-1}\begin{bmatrix} x'_{n-1} \\ y'_{n-1} \\ 1 \end{bmatrix} = z_n\tilde{x}'_{n|1^*} + v_x t_1\tilde{q}_{x|1^*} + v_y t_1\tilde{q}_{y|1^*} + v_z t_1\tilde{q}_{z|1^*} + \tilde{\eta}_{1|1^*} \tag{17}$$

Components of the interest point location, x'_{n-1} and y'_{n-1} can be separately computed using the above equation. If we represent the individual components of a quaternion with $[q]_s$, $[q]_a$, $[q]_b$ and $[q]_c$, x'_{n-1} and y'_{n-1} can be computed as below [1].

$$x'_{n-1} = \frac{z_n[\tilde{x}'_{n|1^*}]_a + v_x t_1[\tilde{q}_{x|1^*}]_a + v_y t_1[\tilde{q}_{y|1^*}]_a + v_z t_1[\tilde{q}_{z|1^*}]_a + [\tilde{\eta}_{1|1^*}]_a}{z_n[\tilde{x}'_{n|1^*}]_c + v_x t_1[\tilde{q}_{x|1^*}]_c + v_y t_1[\tilde{q}_{y|1^*}]_c + v_z t_1[\tilde{q}_{z|1^*}]_c + [\tilde{\eta}_{1|1^*}]_c} \tag{18}$$

$$y'_{n-1} = \frac{z_n[\tilde{x}'_{n|1^*}]_b + v_x t_1[\tilde{q}_{x|1^*}]_b + v_y t_1[\tilde{q}_{y|1^*}]_b + v_z t_1[\tilde{q}_{z|1^*}]_b + [\tilde{\eta}_{1|1^*}]_b}{z_n[\tilde{x}'_{n|1^*}]_c + v_x t_1[\tilde{q}_{x|1^*}]_c + v_y t_1[\tilde{q}_{y|1^*}]_c + v_z t_1[\tilde{q}_{z|1^*}]_c + [\tilde{\eta}_{1|1^*}]_c} \tag{19}$$

Similarly, for x'_{n-2} and y'_{n-2}:

$$x'_{n-2} = \frac{z_n[\tilde{x}'_{n|2^*}]_a + v_x t_2[\tilde{q}_{x|2^*}]_a + v_y t_2[\tilde{q}_{y|2^*}]_a + v_z t_2[\tilde{q}_{z|2^*}]_a + [\tilde{\eta}_{2|2^*}]_a}{z_n[\tilde{x}'_{n|2^*}]_c + v_x t_2[\tilde{q}_{x|2^*}]_c + v_y t_2[\tilde{q}_{y|2^*}]_c + v_z t_2[\tilde{q}_{z|2^*}]_c + [\tilde{\eta}_{2|2^*}]_c} \tag{20}$$

$$y'_{n-2} = \frac{z_n[\tilde{x}'_{n|2^*}]_b + v_x t_2[\tilde{q}_{x|2^*}]_b + v_y t_2[\tilde{q}_{y|2^*}]_b + v_z t_2[\tilde{q}_{z|2^*}]_b + [\tilde{\eta}_{2|2^*}]_b}{z_n[\tilde{x}'_{n|2^*}]_c + v_x t_2[\tilde{q}_{x|2^*}]_c + v_y t_2[\tilde{q}_{y|2^*}]_c + v_z t_2[\tilde{q}_{z|2^*}]_c + [\tilde{\eta}_{2|2^*}]_c} \tag{21}$$

After some primitive algebraic manipulations, the four equations above become:

$$\begin{aligned} v_x\Big(t_1(x'_{n-1}[\tilde{q}_{x|1^*}]_c - [\tilde{q}_{x|1^*}]_a)\Big)+ \\ v_y\Big(t_1(x'_{n-1}[\tilde{q}_{y|1^*}]_c - [\tilde{q}_{y|1^*}]_a)\Big)+ \\ v_z\Big(t_1(x'_{n-1}[\tilde{q}_{z|1^*}]_c - [\tilde{q}_{z|1^*}]_a)\Big)+ \\ z_n\Big(x'_{n-1}[\tilde{x}'_{n|1^*}]_c - [\tilde{x}'_{n|1^*}]_a\Big) = [\tilde{\eta}_{1|1^*}]_a - x'_{n-1}[\tilde{\eta}_{1|1^*}]_c \end{aligned} \tag{22}$$

$$\begin{aligned} v_x\Big(t_1(y'_{n-1}[\tilde{q}_{x|1^*}]_c - [\tilde{q}_{x|1^*}]_b)\Big)+ \\ v_y\Big(t_1(y'_{n-1}[\tilde{q}_{y|1^*}]_c - [\tilde{q}_{y|1^*}]_b)\Big)+ \\ v_z\Big(t_1(y'_{n-1}[\tilde{q}_{z|1^*}]_c - [\tilde{q}_{z|1^*}]_b)\Big)+ \\ z_n\Big(y'_{n-1}[\tilde{x}'_{n|1^*}]_c - [\tilde{x}'_{n|1^*}]_b\Big) = [\tilde{\eta}_{1|1^*}]_b - y'_{n-1}[\tilde{\eta}_{1|1^*}]_c \end{aligned} \tag{23}$$

$$\begin{aligned} v_x\Big(t_2(x'_{n-2}[\tilde{q}_{x|2^*}]_c - [\tilde{q}_{x|2^*}]_a)\Big)+ \\ v_y\Big(t_2(x'_{n-2}[\tilde{q}_{y|2^*}]_c - [\tilde{q}_{y|2^*}]_a)\Big)+ \\ v_z\Big(t_2(x'_{n-2}[\tilde{q}_{z|2^*}]_c - [\tilde{q}_{z|2^*}]_a)\Big)+ \\ z_n\Big(x'_{n-2}[\tilde{x}'_{n|2^*}]_c - [\tilde{x}'_{n|2^*}]_a\Big) = [\tilde{\eta}_{2|2^*}]_a - x'_{n-2}[\tilde{\eta}_{2|2^*}]_c \end{aligned} \tag{24}$$

$$
\begin{aligned}
&v_x\Big(t_2(y'_{n-2}[\tilde{q}_{x|2^*}]_c - [\tilde{q}_{x|2^*}]_b)\Big)+\\
&\quad v_y\Big(t_2(y'_{n-2}[\tilde{q}_{y|2^*}]_c - [\tilde{q}_{y|2^*}]_b)\Big)+\\
&\qquad v_z\Big(t_2(y'_{n-2}[\tilde{q}_{z|2^*}]_c - [\tilde{q}_{z|2^*}]_b)\Big)+\\
&\qquad\quad z_n\Big(y'_{n-2}[\tilde{x}'_{n|2^*}]_c - [\tilde{x}'_{n|2^*}]_b\Big) = [\tilde{\eta}_{2|2^*}]_b - y'_{n-2}[\tilde{\eta}_{2|2^*}]_c
\end{aligned} \tag{25}
$$

Let us define a matrix D, with elements d_{11} - d_{44} and a column matrix $\vec{f}$ with elements f_1 - f_4. We can rewrite (22)-(25) using the newly introduced variables as:

$$\text{Eq. (22): } v_x d_{11} + v_y d_{12} + v_z d_{13} + z_n d_{14} = f_1 \tag{26a}$$

$$\text{Eq. (23): } v_x d_{21} + v_y d_{22} + v_z d_{23} + z_n d_{24} = f_2 \tag{26b}$$

$$\text{Eq. (24): } v_x d_{13} + v_y d_{23} + v_z d_{33} + z_n d_{34} = f_3 \tag{26c}$$

$$\text{Eq. (25): } v_x d_{41} + v_y d_{42} + v_z d_{43} + z_n d_{44} = f_4 \tag{26d}$$

which results in:

$$D \begin{bmatrix} v_x \\ v_y \\ v_z \\ z_n \end{bmatrix} = f \tag{27}$$

Finally, we have derived the equation for the unknown velocities [1]:

$$\vec{v}_e = \begin{bmatrix} v_x \\ v_y \\ v_z \\ z_n \end{bmatrix} = D^{-1} f \tag{28}$$

3. UNCERTAINTY ON THE ESTIMATED VELOCITY

The velocity estimation takes 20 input arguments. The input vector can be written as:

$$\vec{o} = \left[\vec{x}_n'^T \; \vec{x}_{n-1}'^T \; \vec{x}_{n-2}'^T \; \vec{q}_1^T \; \vec{q}_2^T \; \vec{\alpha}_{n-1}^T \; \vec{\alpha}_{n-1}^T\right]^T \tag{29}$$

Following the definition of $\vec{o}$, the covariance matrix O is defined as:

$$O = \begin{bmatrix} \Sigma_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Sigma_{n-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Sigma_{n-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{n-2} \end{bmatrix} \tag{30}$$

Q_1 and Q_2 matrices represent the covariance of the relative attitudes $\tilde{q}_1$ and $\tilde{q}_2$. If we define the velocity estimator function as $\vec{v}_e = l(\vec{o})$, $\vec{v}_e$ being the vector containing the velocities and z_n, the covariance matrix of $\vec{v}_e$ is computed as:

$$V_n = J_{l(\vec{o})}(\vec{o}) O J_{l(\vec{o})}(\vec{o})^T \tag{31}$$

Now, we should find derivatives of $l(\vec{o})$ with respect to the elements of $\vec{o}$.

For demonstration, let us find $\frac{\partial l(\vec{o})}{\partial x'_n}$. We use the equivalence[1] $\frac{\partial D^{-1}}{\partial x'_n} = D^{-1}\frac{\partial D}{\partial x'_n}D^{-1}$.

$$\frac{\partial l(\vec{o})}{\partial x'_n} = \frac{\partial D^{-1}\vec{f}}{\partial x'_n} \tag{32a}$$

$$= \frac{\partial D^{-1}}{\partial x'_n}\vec{f} + D^{-1}\frac{\partial \vec{f}}{\partial x'_n} \tag{32b}$$

$$= D^{-1}\frac{\partial D}{\partial x'_n}D^{-1}\vec{f} + D^{-1}\frac{\partial \vec{f}}{\partial x'_n} \tag{32c}$$

$$= D^{-1}\frac{\partial D}{\partial x'_n}\vec{v}_e + D^{-1}\frac{\partial \vec{f}}{\partial x'_n} \tag{32d}$$

This requires the computation of derivative of D and $\vec{f}$ for each parameter. After these derivatives are computed as shown in Appendix A, columns of the Jacobian matrix is found by using the formula in (32d). The constructed Jacobian matrix is plugged into 31 to find V_e. Upper left 3×3 part of V_e corresponds to the covariance matrix of v_n, V_n.

A. JACOBIAN'S TO BE USED IN UNCERTAINTY ESTIMATION

We need to find derivatives of D and $\vec{f}$ with respect to the elements of the input vector. In this appendix, we present these derivatives.

Let us firstly define the derivative of a rotation operation with respect to quaternion parameters. Rotating a point with a quaternion is conducted as:

$$P' = qPq^{-1} = RP \tag{33}$$

(34) shows the equivalent rotation matrix. The derivatives of P' with respect to the quaternion parameters q_s, q_a, q_b and q_c are presented in (35) - (38). We will denote the quaternion representing $\frac{\partial P'}{\partial q_s}$ as $\tilde{J}_{P'}(q_s)$.

$$\begin{matrix} \tilde{P}' = \tilde{q}\tilde{P}\tilde{q}^{-1} \\ p' = Rp \end{matrix} \Rightarrow R = \begin{bmatrix} q_s^2 + q_a^2 - q_b^2 - q_c^2 & -2q_sq_c + 2q_aq_b & 2q_sq_b + 2q_aq_c \\ 2q_sq_c + 2q_aq_b & q_s^2 - q_a^2 + q_b^2 - q_c^2 & -2q_sq_a + 2q_bq_c \\ -2q_sq_b + 2q_aq_c & 2q_sq_a + 2q_bq_c & q_s^2 - q_a^2 - q_b^2 + q_c^2 \end{bmatrix} \tag{34}$$

If we want to take the Jacobian of the rotation $\tilde{P}' = \tilde{q}\tilde{P}\tilde{q}^{-1}$ with respect to $\tilde{q}$, we have to define $\frac{\partial P'}{\partial q_\xi}$ for $\xi = s, a, b, c$ individually:

$$\frac{\partial P'}{\partial q_s} = \begin{bmatrix} 2q_s & -2q_c & 2q_b \\ 2q_c & 2q_s & -2q_a \\ -2q_b & 2q_a & 2q_s \end{bmatrix} P = R_{s,\tilde{q}}P \tag{35}$$

$$\frac{\partial P'}{\partial q_a} = \begin{bmatrix} 2q_a & 2q_b & 2q_c \\ 2q_b & -2q_a & -2q_s \\ 2q_c & 2q_s & -2q_a \end{bmatrix} P = R_{a,\tilde{q}}P \tag{36}$$

$$\frac{\partial P'}{\partial q_b} = \begin{bmatrix} -2q_b & 2q_a & 2q_s \\ 2q_a & 2q_b & 2q_c \\ -2q_s & 2q_c & -2q_b \end{bmatrix} P = R_{b,\tilde{q}}P \tag{37}$$

$$\frac{\partial P'}{\partial q_c} = \begin{bmatrix} -2q_c & -2q_s & 2q_a \\ 2q_s & -2q_c & 2q_b \\ 2q_a & 2q_b & 2q_c \end{bmatrix} P = R_{c,\tilde{q}}P \tag{38}$$

Now, the partial derivatives are presented.

[1] One can easily derive this equivalence by evaluating the derivative of $D^{-1}D = I$.

$$\frac{\partial D}{\partial x'_n} = \begin{bmatrix} 0 & 0 & 0 & d_{11}/t_1 \\ 0 & 0 & 0 & d_{21}/t_1 \\ 0 & 0 & 0 & d_{31}/t_1 \\ 0 & 0 & 0 & d_{41}/t_1 \end{bmatrix} \tag{39}$$

$$\frac{\partial D}{\partial y'_n} = \begin{bmatrix} 0 & 0 & 0 & d_{12}/t_1 \\ 0 & 0 & 0 & d_{22}/t_1 \\ 0 & 0 & 0 & d_{32}/t_1 \\ 0 & 0 & 0 & d_{42}/t_1 \end{bmatrix} \tag{40}$$

$$\frac{\partial D}{\partial x'_{n-1}} = \begin{bmatrix} t_1[\tilde{q}_{x|1}]_c & t_1[\tilde{q}_{y|1}]_c & t_1[\tilde{q}_{z|1}]_c & [\vec{x}_{n|1}]_c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{41}$$

$$\frac{\partial D}{\partial y'_{n-1}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ t_1[\tilde{q}_{x|1}]_c & t_1[\tilde{q}_{y|1}]_c & t_1[\tilde{q}_{z|1}]_c & [\vec{x}_{n|1}]_c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{42}$$

$$\frac{\partial D}{\partial x'_{n-2}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ t_2[\tilde{q}_{x|2}]_c & t_2[\tilde{q}_{y|2}]_c & t_2[\tilde{q}_{z|2}]_c & [\vec{x}_{n|2}]_c \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{43}$$

$$\frac{\partial D}{\partial y'_{n-2}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ t_2[\tilde{q}_{x|2}]_c & t_2[\tilde{q}_{y|2}]_c & t_2[\tilde{q}_{z|2}]_c & [\vec{x}_{n|2}]_c \end{bmatrix} \tag{44}$$

For $\xi = \{s, a, b, c\}$,

$$\frac{\partial D}{\partial q'_{1,\xi}} =$$
$$\begin{bmatrix} t_1(x'_{n-1}[\tilde{J}_{\tilde{q}_{x|1^*}}(q_{1,\xi})]_c - [\tilde{J}_{\tilde{q}_{x|1^*}}(q_{1,\xi})]_a) & t_1(x'_{n-1}[\tilde{J}_{\tilde{q}_{y|1^*}}(q_{1,\xi})]_c - [\tilde{J}_{\tilde{q}_{y|1^*}}(q_{1,\xi})]_a) \\ t_1(y'_{n-1}[\tilde{J}_{\tilde{q}_{x|1^*}}(q_{1,\xi})]_c - [\tilde{J}_{\tilde{q}_{x|1^*}}(q_{1,\xi})]_b) & t_1(y'_{n-1}[\tilde{J}_{\tilde{q}_{y|1^*}}(q_{1,\xi})]_c - [\tilde{J}_{\tilde{q}_{y|1^*}}(q_{1,\xi})]_b) \\ 0 & 0 \\ 0 & 0 \end{bmatrix.$$
$$\begin{matrix} t_1(x'_{n-1}[\tilde{J}_{\tilde{q}_{z|1^*}}(q_{1,\xi})]_c - [\tilde{J}_{\tilde{q}_{z|1^*}}(q_{1,\xi})]_a) & x'_{n-1}[\tilde{J}_{\vec{x}_{n|1^*}}(q_{1,\xi})]_c - [\tilde{J}_{\vec{x}_{n|1^*}}(q_{1,\xi})]_a \\ t_1(y'_{n-1}[\tilde{J}_{\tilde{q}_{z|1^*}}(q_{1,\xi})]_c - [\tilde{J}_{\tilde{q}_{z|1^*}}(q_{1,\xi})]_b) & y'_{n-1}[\tilde{J}_{\vec{x}_{n|1^*}}(q_{1,\xi})]_c - [\tilde{J}_{\vec{x}_{n|1^*}}(q_{1,\xi})]_b \\ 0 & 0 \\ 0 & 0 \end{matrix}\Bigg] \tag{45}$$

$$\frac{\partial D}{\partial q'_{2,\xi}} =$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ t_2(x'_{n-2}[\tilde{J}_{\tilde{q}_{x|2^*}}(q_{2,\xi})]_c - [\tilde{J}_{\tilde{q}_{x|2^*}}(q_{2,\xi})]_a) & t_2(x'_{n-2}[\tilde{J}_{\tilde{q}_{y|2^*}}(q_{2,\xi})]_c - [\tilde{J}_{\tilde{q}_{y|2^*}}(q_{2,\xi})]_a) \\ t_2(y'_{n-2}[\tilde{J}_{\tilde{q}_{x|2^*}}(q_{2,\xi})]_c - [\tilde{J}_{\tilde{q}_{x|2^*}}(q_{2,\xi})]_b) & t_2(y'_{n-2}[\tilde{J}_{\tilde{q}_{y|2^*}}(q_{2,\xi})]_c - [\tilde{J}_{\tilde{q}_{y|2^*}}(q_{2,\xi})]_b) \end{bmatrix.$$
$$\begin{matrix} 0 & 0 \\ 0 & 0 \\ t_2(x'_{n-2}[\tilde{J}_{\tilde{q}_{z|2^*}}(q_{2,\xi})]_c - [\tilde{J}_{\tilde{q}_{z|2^*}}(q_{2,\xi})]_a) & x'_{n-2}[\tilde{J}_{\vec{x}_{n|2^*}}(q_{2,\xi})]_c - [\tilde{J}_{\vec{x}_{n|2^*}}(q_{2,\xi})]_a \\ t_2(y'_{n-2}[\tilde{J}_{\tilde{q}_{z|2^*}}(q_{2,\xi})]_c - [\tilde{J}_{\tilde{q}_{z|2^*}}(q_{2,\xi})]_b) & y'_{n-2}[\tilde{J}_{\vec{x}_{n|2^*}}(q_{2,\xi})]_c - [\tilde{J}_{\vec{x}_{n|2^*}}(q_{2,\xi})]_b \end{matrix}\Bigg] \tag{46}$$

$$\frac{\partial D}{\partial \alpha'_{(n-1,x)}} = \frac{\partial D}{\partial \alpha'_{(n-1,y)}} = \frac{\partial D}{\partial \alpha'_{(n-1,z)}} = \frac{\partial D}{\partial \alpha'_{(n-2,x)}} = \frac{\partial D}{\partial \alpha'_{(n-2,y)}} = \frac{\partial D}{\partial \alpha'_{(n-2,z)}} = 0 \tag{47}$$

$$\frac{\partial \vec{f}}{\partial x'_n} = \frac{\partial \vec{f}}{\partial y'_n} = 0 \tag{48}$$

$$\frac{\partial \vec{f}}{\partial x'_{n-1}} = \begin{bmatrix} -[\vec{\eta}_{1|1^*}]_c \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{49}$$

$$\frac{\partial \vec{f}}{\partial y'_{n-1}} = \begin{bmatrix} 0 \\ -[\vec{\eta}_{1|1^*}]_c \\ 0 \\ 0 \end{bmatrix} \tag{50}$$

$$\frac{\partial \vec{f}}{\partial x'_{n-2}} = \begin{bmatrix} 0 \\ 0 \\ -[\vec{\eta}_{2|2^*}]_c \\ 0 \end{bmatrix} \tag{51}$$

$$\frac{\partial \vec{f}}{\partial y'_{n-2}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -[\vec{\eta}_{2|2^*}]_c \end{bmatrix} \tag{52}$$

For $\xi = \{s, a, b, c\}$,

$$\frac{\partial \vec{f}}{\partial q'_{1,\xi}} = \begin{bmatrix} 0 \\ 0 \\ -(\frac{1}{2}t_1^2 + t_1 t_3)([\tilde{q}_2^* \tilde{J}_{\alpha_{n-1}}(q_{1,\xi})\tilde{q}_2]_a - x_{n-2}[\tilde{q}_2^* \tilde{J}_{\alpha_{n-1}}(q_{1,\xi})\tilde{q}_2]_c) \\ -(\frac{1}{2}t_1^2 + t_1 t_3)([\tilde{q}_2^* \tilde{J}_{\alpha_{n-1}}(q_{1,\xi})\tilde{q}_2]_b - y_{n-2}[\tilde{q}_2^* \tilde{J}_{\alpha_{n-1}}(q_{1,\xi})\tilde{q}_2]_c) \end{bmatrix} \tag{53}$$

$$\frac{\partial \vec{f}}{\partial q'_{2,\xi}} = \begin{bmatrix} 0 \\ 0 \\ -(\frac{1}{2}t_1^2 + t_1 t_3)([\tilde{J}_{\alpha_{n-1|1}}(q_{2,s})]_a - x_{n-2}[\tilde{J}_{\alpha_{n-1|1}}(q_{2,s})]_c) \\ -(\frac{1}{2}t_1^2 + t_1 t_3)([\tilde{J}_{\alpha_{n-1|1}}(q_{2,s})]_b - y_{n-2}[\tilde{J}_{\alpha_{n-1|1}}(q_{2,s})]_c) \end{bmatrix} \tag{54}$$

$$\frac{\partial \vec{f}}{\partial \alpha'_{(n-1,x)}} \begin{bmatrix} -\frac{1}{2}t_1^2 \\ 0 \\ ([q_{x|3^*}]_a - x'_{n-2}[q_{x|3^*}]_c)(\frac{1}{2}t_1^2 + t_1 t_3) \\ ([q_{x|3^*}]_b - x'_{n-2}[q_{x|3^*}]_c)(\frac{1}{2}t_1^2 + t_1 t_3) \end{bmatrix} \tag{55}$$

$$\frac{\partial \vec{f}}{\partial \alpha'_{(n-1,y)}} \begin{bmatrix} 0 \\ -\frac{1}{2}t_1^2 \\ ([q_{y|3^*}]_a - x'_{n-2}[q_{y|3^*}]_c)(\frac{1}{2}t_1^2 + t_1 t_3) \\ ([q_{y|3^*}]_b - x'_{n-2}[q_{y|3^*}]_c)(\frac{1}{2}t_1^2 + t_1 t_3) \end{bmatrix} \tag{56}$$

$$\frac{\partial \vec{f}}{\partial \alpha'_{(n-1,z)}} \begin{bmatrix} \frac{1}{2}t_1^2 x_{n-1} \\ \frac{1}{2}t_1^2 y_{n-1} \\ ([q_{z|3^*}]_a - x'_{n-2}[q_{z|3^*}]_c)(\frac{1}{2}t_1^2 + t_1 t_3) \\ ([q_{z|3^*}]_b - x'_{n-2}[q_{z|3^*}]_c)(\frac{1}{2}t_1^2 + t_1 t_3) \end{bmatrix} \tag{57}$$

$$\frac{\partial \vec{f}}{\partial \alpha'_{(n-2,x)}} \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}t_3^2 \\ 0 \end{bmatrix} \tag{58}$$

$$\frac{\partial \vec{f}}{\partial \alpha'_{(n-2,y)}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2}t_3^2 \end{bmatrix} \tag{59}$$

$$\frac{\partial \vec{f}}{\partial \alpha'_{(n-2,z)}} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}t_3^2 x_{n-2} \\ \frac{1}{2}t_3^2 y_{n-2} \end{bmatrix} \tag{60}$$

B. REFERENCES

[1] Laurent Kneip, Agostino Martinelli, Stephan Weiss, Davide Scaramuzza, and Roland Siegwart, "Closed-form solution for absolute scale velocity determination combining inertial measurements and a single feature correspondence," in *IEEE International Conference on Robotics and Automation (ICRA)*, 2011.